

# Harmonic knots

P. -V. Koseleff, D. Pecker

Université Pierre et Marie Curie  
4, place Jussieu, F-75252 Paris Cedex 05  
e-mail: {koseleff,pecker}@math.jussieu.fr

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## Abstract

The harmonic knot  $H(a, b, c)$  is parameterized as  $K(t) = (T_a(t), T_b(t), T_c(t))$  where  $a, b$  and  $c$  are relatively coprime integers and  $T_n$  is the degree  $n$  Chebyshev polynomial of the first kind. We classify the harmonic knots  $H(a, b, c)$  for  $a \leq 4$ . We show that the knot  $H(2n-1, 2n, 2n+1)$  is isotopic to  $H(4, 2n-1, 2n+1)$  (up to mirror symmetry). We study the knots  $H(5, n, n+1)$  and give a table of the simplest harmonic knots.

**keywords:** Polynomial curves, Chebyshev curves, rational knots, continued fractions

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## 1 Introduction

A harmonic curve (or Chebyshev curve) is defined to be a curve which admits a parametrization  $x = T_a(t)$ ,  $y = T_b(t)$ ,  $z = T_c(t)$  where  $a$ ,  $b$  and  $c$  are relatively coprime integers and  $T_n(t)$  are the classical Chebyshev polynomials defined by  $T_n(\cos t) = \cos nt$ . A harmonic knot is a non singular harmonic curve. In 1897 Comstock proved that a harmonic curve is a knot if and only if  $a, b, c$  are pairwise coprime integers ([Com, KP3, FF]).

We observed in [KP1] that the trefoil can be parametrized by Chebyshev polynomials:  $x = T_3(t)$ ;  $y = T_4(t)$ ;  $z = T_5(t)$ . This led us to study harmonic knots in [KP3].

Harmonic knots are polynomial analogues of the famous Lissajous knots ([BDHZ, BHJS, Cr, HZ, JP, La1, La2]). The symmetries of harmonic knots, obvious from the parity of Chebyshev polynomials, are different from those of Lissajous. For example, the figure-eight knot which is not a Lissajous knot, is the harmonic knot  $H(3, 5, 7)$ .

We proved that the harmonic knot  $H(a, b, ab - a - b)$  is alternating, and deduced that there are infinitely many amphicheiral harmonic knots and infinitely many strongly invertible harmonic knots. We also proved in [KP3] that the torus knot  $T(2, 2n + 1)$  is the harmonic knot  $H(3, 3n + 2, 3n + 1)$ .

The harmonic knots  $H(3, b, c)$  are classified in [KP4]; they are two-bridge knots and their Schubert fractions  $\frac{\alpha}{\beta}$  verify  $\beta^2 \equiv \pm 1 \pmod{\alpha}$ .

In this article, we give the classification of the harmonic knots  $H(4, b, c)$  for  $b$  and  $c$  coprime odd integers. We also study some infinite families of harmonic knots for  $a \geq 5$ .

In section 2 we recall the Conway notation for two-bridge knots, and the computation of their Schubert fractions with continued fractions. We observe that the knots  $H(4, b, c)$  are two-bridge knots, and their Schubert fractions are given by continued fractions of the form  $[\pm 1, \pm 2, \dots, \pm 1, \pm 2]$ . We show results on these continued fraction expansions. In section 3 we compute the Schubert fractions of the harmonic knots  $H(4, b, c)$ . We deduce the classification of these knots:

**Theorem 3.7.** *Let  $b$  and  $c$  be relatively prime odd integers, and let  $K = H(4, b, c)$ . There is a unique pair  $(b', c')$  such that (up to mirroring)*

$$K = H(4, b', c'), \quad b' < c' < 3b', \quad b' \not\equiv c' \pmod{4}.$$

*$K$  has a Schubert fraction  $\frac{\alpha}{\beta}$  such that  $\beta^2 \equiv \pm 2 \pmod{\alpha}$ . Furthermore, there is an algorithm to find  $(b', c')$ , and the crossing number of  $K$  is  $N = (3b' + c' - 2)/4$ .*

We notice that the trefoil is the only knot which is both of form  $H(3, b, c)$  and  $H(4, b, c)$ . In section 4 we study some families of harmonic knots  $H(a, b, c)$  with  $a \geq 5$ . In general the bridge number of these knots is greater than two, this is why the following result is surprising.

**Theorem 4.5.** *The harmonic knot  $H(2n - 1, 2n, 2n + 1)$  is isotopic to the two-bridge harmonic knot  $H(4, 2n - 1, 2n + 1)$ , up to mirror symmetry.*

We also find an infinite family of two-bridge harmonic knots which are not of the form  $H(a, b, c)$  for  $a \leq 4$ :

**Theorem 4.6.**

*The knot  $H(5, 5n + 1, 5n + 2)$  is the two-bridge knot of Conway form  $C(2n + 1, 2n)$ .*

*The knot  $H(5, 5n + 3, 5n + 4)$  is the two-bridge knot of Conway form  $C(2n + 1, 2n + 2)$ .*

*Except for  $H(5, 6, 7) = H(4, 5, 7)$  and  $H(5, 3, 4)$ , these knots are not of the form  $H(a, b, c)$  with  $a \leq 4$ .*

Then, we give an example of a composite harmonic knot, which disproves a conjecture of Freudenburg and Freudenburg [FF]. We show that the nonreversible knot  $8_{17}$  is a harmonic knot.

Then, we identify the knots  $H(a, b, c)$  for  $(a - 1)(b - 1) \leq 30$ . We conclude the paper with some questions and conjectures.

## 2 Continued fractions and rational Chebyshev knots

A two-bridge knot (or link) admits a diagram in Conway's normal form. This form, denoted by  $C(a_1, a_2, \dots, a_n)$  where  $a_i$  are integers, is explained by the following picture (see [Con], [Mu] p. 187). The number of twists is denoted by the integer  $|a_i|$ , and the sign of  $a_i$  is

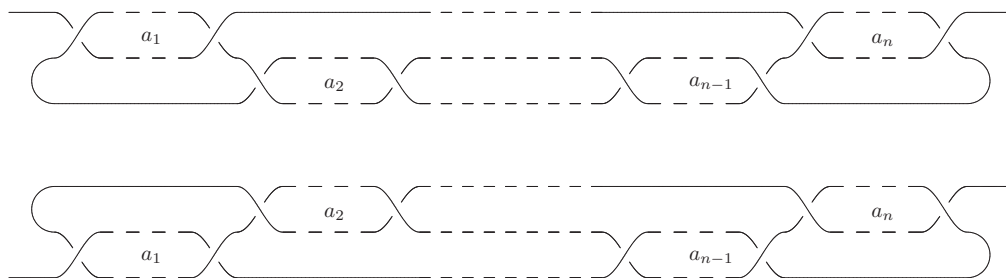


Figure 1: Conway normal forms for polynomial knots ( $n$  odd, and  $n$  even)

defined as follows: if  $i$  is odd, then the right twist is positive, if  $i$  is even, then the right twist is negative. In Figure 1 the  $a_i$  are positive (the  $a_1$  first twists are right twists).

The two-bridge knots (or links) are classified by their Schubert fractions

$$\frac{\alpha}{\beta} = a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}} = [a_1, \dots, a_n], \quad \alpha > 0.$$

We shall denote  $S(\frac{\alpha}{\beta})$  a two-bridge knot (or link) with Schubert fraction  $\frac{\alpha}{\beta}$ . The two-bridge knots (or links)  $S(\frac{\alpha}{\beta})$  and  $S(\frac{\alpha'}{\beta'})$  are equivalent if and only if  $\alpha = \alpha'$  and  $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$ . If  $K = S(\frac{\alpha}{\beta})$ , its mirror image is  $\overline{K} = S(\frac{\alpha}{-\beta})$ .

We shall study knots with a diagram illustrated by figure 2. In this case, the  $a_i$  and the

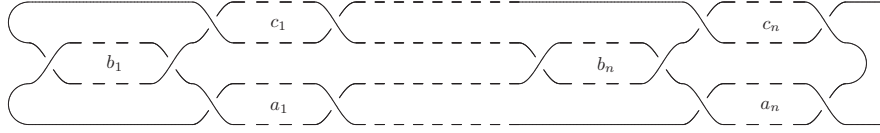


Figure 2: A knot isotopic to  $C(b_1, a_1 + c_1, b_2, a_2 + c_2, \dots, b_n, a_n + c_n)$

$c_i$  are positive if they are left twists, the  $b_i$  are positive if they are right twists (in our figure  $a_i, b_i, c_i$  are positive). Such a knot is equivalent to a knot with Conway's normal form  $C(b_1, a_1 + c_1, b_2, a_2 + c_2, \dots, b_n, a_n + c_n)$  (see [Mu] p. 183-184). Many of our knots have a Chebyshev diagram  $\mathcal{C}(4, k) : x = T_4(t), y = T_k(t)$ . In this case we obtain diagrams of the form illustrated by Figure 2. Consequently, such a knot has a Schubert fraction of the form  $[b_1, d_1, b_2, d_2, \dots, b_n, d_n]$  with  $b_i = \pm 1$ ,  $d_i = \pm 2$  or  $d_i = 0$ .

Once again, the situation is best explained by typical examples. Figure 3 represents two knots with the same Chebyshev diagram  $\mathcal{C}(4, 5) : x = T_4(t), y = T_5(t)$ . A Schubert fraction of the first knot is  $\frac{5}{2} = [1, 0, 1, 2]$ ; it is the figure-eight knot. A Schubert fraction of the second knot is  $\frac{7}{-4} = [-1, -2, 1, 2]$ ; it is the twist knot  $5_2$ . By symmetry, the Chebyshev diagrams

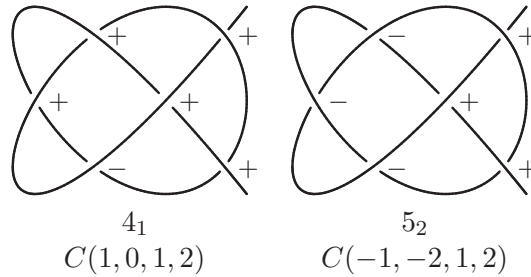


Figure 3: Knots with the Chebyshev diagram  $\mathcal{C}(4, 5)$

of harmonic knots  $H(4, b, c)$  are of Conway form  $C(\pm 1, \pm 2, \dots, \pm 1, \pm 2)$ . Consequently, the diagram  $C(1, 0, 1, 2)$  is not the Chebyshev diagram of a harmonic knot  $H(4, b, c)$ .

## 2.1 Continued fractions

Let  $K$  be a two-bridge knot defined by a continued fraction expansion of its Schubert fraction  $\frac{\alpha}{\beta} = [q_1, q_2, \dots, q_n]$ , where the  $q_i$  are not necessarily positive. It is often possible to

obtain directly the crossing number of  $K$ .

**Definition 2.1.** Let  $r > 0$  be a rational number, and  $r = [q_1, \dots, q_n]$  be a continued fraction with  $q_i > 0$ . The crossing number of  $r$  is defined by  $\text{cn}(r) = q_1 + \dots + q_n$ .

The following result is proved in [KP4].

**Proposition 2.2.** Let  $\frac{\alpha}{\beta} = [a_1, \dots, a_n]$  be a continued fraction such that  $a_1 a_2 > 0$ ,  $a_{n-1} a_n > 0$ , and there is no two consecutive sign changes in the sequence  $a_1, a_2, \dots, a_n$ . Then its crossing number is

$$\text{cn}\left(\frac{\alpha}{\beta}\right) = \sum_{k=1}^n |a_k| - \#\{j, a_j a_{j+1} < 0\}. \quad (1)$$

## 2.2 Continued fractions $[\pm 1, \pm 2, \dots, \pm 1, \pm 2]$

We begin with a useful lemma:

**Lemma 2.3.** Let  $r = [1, 2e_2, e_3, 2e_4, \dots, e_{2m-1}, 2e_{2m}]$ ,  $e_i = \pm 1$ . We suppose that there are no three consecutive sign changes in the sequence  $e_1, \dots, e_{2m}$ . Then  $r > 0$ , and  $r > 1$  if and only if  $e_2 = 1$ .

*Proof.* By induction on  $m$ .

If  $m = 1$ , then  $r = [1, 2] = \frac{3}{2}$  or  $r = [1, -2] = \frac{1}{2}$ , and the result is true.

Suppose the result true for  $m - 1$ , and let us prove it for  $m$ .

First, let us suppose  $r = [1, 2, e_3, \dots, 2e_{2m}]$ .

If  $e_3 = 1$ , then  $r = [1, 2, y] = \frac{3y+1}{2y+1}$ , where  $y = [1, \pm 2, \dots]$ . By induction we have  $y > 0$ , and then  $r > 1$ .

If  $e_3 = -1$  and  $e_4 = 1$ . Then  $e_5 = 1$  and  $r = [1, 2, -1, 2, y] = y + 2$  with  $y = [1, \pm 2, \dots]$ . We have  $y > 0$  by induction, and then  $r > 2 > 1$ .

If  $e_3 = e_4 = -1$ , then  $r = [1, 2, -y] = \frac{3y-1}{2y-1} = \frac{3}{2} + \frac{1}{2(2y-1)}$  with  $y = [1, 2, \pm 1, \dots]$ . We have  $y > 1$  by induction, and then  $r > \frac{3}{2} > 1$ .

Now, let us suppose  $r = [1, -2, \dots]$ .

If  $r = [1, -2, -1, \dots]$ . Then  $r = [1, -2, -y] = \frac{y+1}{2y+1}$ , with  $y = [1, \pm 2, \dots]$ . By induction, we have  $y > 0$ , and then  $0 < r < 1$ .

If  $r = [1, -2, 1, \dots]$ . Then  $r = [1, -2, 1, 2, \dots] = [1, -2, y] = \frac{y-1}{2y-1}$  where  $y = [1, 2, \pm 1, \dots]$ . By induction we have  $y > 1$ , and then  $0 < r < 1$ .

This completes the proof.  $\square$

**Remark 2.4.** Because of the identities  $x = [1, -2, 1, -2, x]$  and  $x = [2, -1, 2, -1, x]$ , we see that the condition on the sign changes is necessary. It is also necessary in the following theorem.

**Theorem 2.5.** Let  $r = \frac{\alpha}{\beta} > 0$  be a fraction with  $\alpha$  odd and  $\beta$  even. There is a unique continued fraction expansion  $r = [1, \pm 2, \dots, \pm 1, \pm 2]$  without three consecutive sign changes.

*Proof.* The existence of this continued fraction expansion is given in [KPR]. Its uniqueness is a direct consequence of lemma 2.3.  $\square$

The next result will be useful to describe the continued fractions of harmonic knots  $H(4, b, c)$ .

**Proposition 2.6.** Let  $r = \frac{\alpha}{\beta}$  be a rational number given by a continued fraction of the form  $r = [e_1, 2e_2, e_3, 2e_4, \dots, e_{2m-1}, 2e_{2m}]$ ,  $e_1 = 1$ ,  $e_i = \pm 1$ . We suppose that the sequence of sign changes is palindromic, that is  $e_k e_{k+1} = e_{2m-k} e_{2m-k+1}$  for  $k = 1, \dots, 2m-1$ .

Then we have  $\beta^2 \equiv \pm 2 \pmod{\alpha}$ .

*Proof.* We shall use the Möbius transformations

$$A(x) = [1, x] = \frac{x+1}{x+0}, \quad B(x) = [2, x] = \frac{2x+1}{x+0}, \quad S(x) = -x$$

and their matrix notations

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad ASB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

We shall consider the mapping (analogous to matrix transposition)

$$\tau : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & \frac{c}{2b} \\ 2b & d \end{bmatrix}.$$

We have  $\tau(XY) = \tau(Y)\tau(X)$ ,  $\tau(AB) = AB$ ,  $\tau(ASB) = ASB$  and  $\tau(S) = S$ .

Let  $G$  be the Möbius transformation defined by  $G(z) = [1, 2e_2, e_3, 2e_4, \dots, e_{2m-1}, 2e_{2m}, z]$ , we have  $\frac{\alpha}{\beta} = G(\infty)$ . Let us write  $G = X_1 \cdots X_n$  where  $X_i = A, B$  or  $S$ ,  $X_1 = A$  and  $X_n = B$ . One can suppose that  $G$  contains no subsequence of the form  $AA, ASA, BB, SS$  and  $BSB$ . Moreover, the palindromic condition means that if  $X_i = S$ , then  $X_{n+1-i} = S$ .

Let us show that if  $P = X_1 \cdots X_n$  is a product of terms  $A, B, S$  having these properties, then  $\tau(P) = P$ . By induction on  $s = \#\{i, X_i = S\}$ .

If  $s = 0$  then  $P = (AB)^m$ , and  $\tau(P) = (\tau(AB))^m = (AB)^m = P$ .

Let  $k = \min\{i, X_i = S\}$ .

If  $k = 2q+1$  then  $q \geq 1$  and  $P = (AB)^q S P' S (AB)^q$ . By induction we have  $\tau(P') = P'$ , and then  $\tau(P) = \tau((AB)^q) \tau(S) \tau(P') \tau(S) \tau((AB)^q) = P$ .

If  $k = 2q$  then  $P = (AB)^{q-1} (ASB) P' (ASB) (AB)^{q-1}$ . By induction we have  $\tau(P') = P'$ , and then  $\tau(P) = P$ . This concludes our induction proof.

Consequently we have  $\tau(G) = G$ . Since  $G \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ , we see that  $G = \begin{bmatrix} \alpha & \gamma \\ \beta & \lambda \end{bmatrix}$ , with  $\beta = 2\gamma$ . Since  $\det(G) = \pm 1$ , we obtain  $\beta^2 \equiv \pm 2 \pmod{\alpha}$ .  $\square$

### 3 The harmonic knots $H(a, b, c)$

We shall first show some properties of the plane Chebyshev curves  $x = T_a(t)$ ,  $y = T_b(t)$ . We shall need the following result proved in [KP3]:

**Proposition 3.1.** *Let  $a$  and  $b$  be coprime integers. The  $\frac{1}{2}(a-1)(b-1)$  double points of the Chebyshev curve  $x = T_a(t)$ ,  $y = T_b(t)$  are obtained for the parameter pairs*

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi, \quad s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi,$$

where  $h, k$  are positive integers such that  $\frac{k}{a} + \frac{h}{b} < 1$ .

Using the symmetries of Chebyshev polynomials, we see that this set of parameters is symmetrical about the origin. Let us define a right twist as in Figure 4(a) and a left twist as in Figure 4(b); this notion depends on the choice of the coordinate axes.

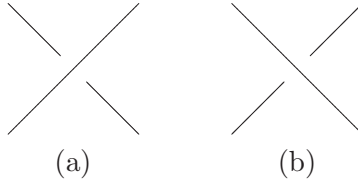


Figure 4: A right twist (a) and a left twist (b).

We will write  $x \sim y$  when  $\text{sign}(x) = \text{sign}(y)$ . We shall need the following result proved in [KP3, KPR].

**Lemma 3.2.** *Let  $H(a, b, c)$  be a harmonic knot.*

*A crossing point of parameter  $t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi$  is a right twist if and only if*

$$D = \left(z(t) - z(s)\right)x'(t)y'(t) > 0$$

where

$$z(t) - z(s) \sim -\sin\left(\frac{ch}{b}\pi\right)\sin\left(\frac{ck}{a}\pi\right) \text{ and } x'(t)y'(t) \sim (-1)^{h+k}\sin\left(\frac{ah}{b}\pi\right)\sin\left(\frac{bk}{a}\pi\right).$$

**Remark 3.3.** The sign of  $\sin(r\pi)$  is simply given by  $(-1)^{\lfloor r \rfloor}$ .

From this lemma we immediately deduce

**Corollary 3.4.** *Let  $a, b, c$  be coprime integers. Suppose that the integer  $c'$  verifies  $c' \equiv c \pmod{2a}$  and  $c' \equiv -c \pmod{2b}$ . Then the knot  $H(a, b, c')$  is the mirror image of  $H(a, b, c)$ .*

*Proof.* At each crossing point we have  $T_{c'}(t) - T_{c'}(s) = -(T_c(t) - T_c(s))$ .  $\square$

**Corollary 3.5.** *Let  $a, b, c$  be coprime integers. Suppose that the integer  $c$  is of the form  $c = \lambda a + \mu b$  with  $\lambda, \mu > 0$ . Then there exists  $c' < c$  such that  $H(a, b, c) = \overline{H}(a, b, c')$*

*Proof.* Let  $c' = |\lambda a - \mu b|$ . The result follows immediately from corollary 3.4  $\square$

This corollary is often used to reduce the degree of a harmonic knot. In a recent paper, G. and J. Freudenburg proved the following stronger result. *There is a polynomial automorphism  $\Phi$  of  $\mathbf{R}^3$  such that  $\Phi(H(a, b, c)) = H(a, b, c')$ .* They also conjectured that for any coprime integers  $a < b$ , the knots  $H(a, b, c)$ ,  $a < b < c$ ,  $c \neq \lambda a + \mu b$ ,  $\lambda, \mu > 0$  are different knots ([FF], Conjecture 6.2).

In [KP4] we obtained the Schubert fractions of the harmonic knots  $H(3, b, c)$ . We deduced their classification, which provides a proof of Freudenburg's conjecture for  $a = 3$ . We shall follow the same strategy to study the harmonic knots  $H(4, b, c)$ .

### 3.1 The harmonic knots $H(4, b, c)$ .

The following result will allow us to classify the harmonic knots of the form  $H(4, b, c)$ .

**Theorem 3.6.** *Let  $b, c$  be odd integers such that  $b \not\equiv c \pmod{4}$ . The Schubert fraction of the knot  $K = H(4, b, c)$  is given by the continued fraction*

$$\frac{\alpha}{\beta} = [e_1, 2e_2, e_3, 2e_4, \dots, e_{b-2}, 2e_{b-1}], \text{ where } e_j = \text{sign}(\sin(j-b)\theta), \theta = \frac{3b-c}{4b}\pi.$$

*We have  $\beta^2 \equiv \pm 2 \pmod{\alpha}$ . If  $b < c < 3b$ , then the crossing number of  $K$  is  $N = (3b+c-2)/4$ .*

The proof will be given in section 3.2, p. 10.

We are now able to classify the harmonic knots of the form  $H(4, b, c)$ .

**Theorem 3.7.** *Let  $b$  and  $c$  be relatively prime odd integers, and let  $K = H(4, b, c)$ . There is a unique pair  $(b', c')$  such that (up to mirroring)*

$$K = H(4, b', c'), \quad b' < c' < 3b', \quad b' \not\equiv c' \pmod{4}.$$

*$K$  has a Schubert fraction  $\frac{\alpha}{\beta}$  such that  $\beta^2 \equiv \pm 2 \pmod{\alpha}$ . Furthermore, there is an algorithm to find  $(b', c')$ , and the crossing number of  $K$  is  $N = (3b' + c' - 2)/4$ .*



*Proof.* Let us first prove the uniqueness of this pair. Let  $K = H(4, b, c)$  with  $b < c < 3b$ ,  $c \not\equiv b \pmod{4}$ . We thus have  $c = 3b \pmod{4}$ , so  $\lambda = \frac{1}{4}(3b - c)$  is an integer and by Theorem 3.6,  $K$  admits the Schubert fraction  $\frac{\alpha}{\beta} = [e_1, 2e_2, e_3, 2e_4, \dots, e_{b-2}, 2e_{b-1}]$  where  $e_j = \text{sign}(\sin(j - b)\theta) = (-1)^\lambda \text{sign}(\sin j\theta)$ .

As  $0 < \theta < \frac{\pi}{2}$ , we deduce that  $e_1$  and  $e_2$  have the same signs. Furthermore  $|\beta| < \alpha$  by Lemma 2.3,  $\beta^2 = \pm 2 \pmod{\alpha}$  by Theorem 3.6 and  $\beta$  is even by Theorem 2.5. We then have  $\alpha \neq 5$ .

Suppose that  $\frac{\alpha}{\beta'}$  is another Schubert fraction of  $K$  with  $|\beta'| \leq \alpha$ ,  $\beta'^2 = \pm 2 \pmod{\alpha}$  and  $\beta'$  even. We then must have  $\beta\beta' \equiv 1 \pmod{\alpha}$  so  $\pm 4 \equiv 1 \pmod{\alpha}$ . We thus deduce that  $\frac{\alpha}{\beta} = \frac{3}{2}$ , and then  $K$  is a trefoil.

In any case  $\frac{\alpha}{\beta}$  is the unique Schubert fraction of  $K$  which satisfies  $|\beta| < \alpha$ ,  $\beta^2 = \pm 2 \pmod{\alpha}$  and  $\beta$  even. The integer  $b - 1$  is then the length of the continued fraction expansion of  $\frac{\alpha}{\beta} = [e_1, 2e_2, e_3, 2e_4, \dots, e_{b-2}, 2e_{b-1}]$ . Since we also have  $3b + c - 2 = 4 \text{cn}(K)$ , we conclude that  $(b, c)$  is uniquely determined by  $K$ .

Now, let us prove the existence of the pair  $(b', c')$ . Let  $K = H(4, b, c)$ ,  $b < c$ . We have only to show that if the pair  $(b, c)$  does not satisfy the condition of the theorem, then it is possible to reduce it.

If  $c \equiv b \pmod{4}$ , then  $c = b + 4\mu$ ,  $\mu > 0$ , and we can reduce the pair  $(b, c)$  by corollary 3.5.

If  $c \not\equiv b \pmod{4}$  and  $c > 3b$ , we have  $c = 3b + 4\mu$ ,  $\mu > 0$ , and we can reduce  $(b, c)$  by 3.5.  $\square$

**Remark 3.8.** Our theorem gives a positive answer to the Freudenburg conjecture ([FF, 6.2]) for  $a = 4$ : *the knots  $H(4, b, c)$ ,  $4 < b < c$ ,  $c \neq 4\lambda + \mu b$ ,  $\lambda, \mu > 0$  are different knots.*

We also see that the only knot belonging to the two families of knots  $H(3, b, c)$  and  $H(4, b, c)$  is the trefoil  $H(3, 4, 5) = \overline{H}(4, 3, 5)$ .

**Corollary 3.9.** *The harmonic knot  $H(4, 2k - 1, 2k + 1)$  is the two-bridge knot of Conway form  $C(3, 2, \dots, 2)$  and crossing number  $2k - 1$ .*

*Proof.* By Theorem 3.6, the knot  $H_k = H(4, 2k - 1, 2k + 1)$  has crossing number  $2k - 1$  and Conway form  $C(e_1, 2e_2, \dots, e_{2k-3}, 2e_{2k-2})$ , where  $e_j = \text{sign}(\sin(j - b)\theta)$ ,  $\theta = \frac{\pi}{2}(1 - \frac{1}{2k-1})$ .

Since the knots  $C(a_1, \dots, a_{2m})$  and  $C(-a_{2m}, \dots, -a_1)$  are isotopic, we deduce that  $H_k$  is isotopic to the knot  $C(2\varepsilon_1, \varepsilon_2, \dots, 2\varepsilon_{2k-3}, \varepsilon_{2k-2})$  where  $\varepsilon_i = \text{sign}(\sin i\theta) = (-1)^{\lfloor \frac{i-1}{2} \rfloor}$ .

We deduce that the rational number  $r_k = [2, 1, -2, -1, \dots, (-1)^{k-2}2, (-1)^{k-2}]$  (length  $2k - 2$ ) is a Schubert fraction of  $H_k$ . We have  $r_2 = 3$ , and  $r_k = [2, 1, -r_{k-1}]$ . Using the identity  $[2, 1, x] = [3, x - 1]$ , by an easy induction we obtain  $r_k = [3, 2, \dots, 2]$ .  $\square$

**Example 3.10 (The Twist knots).** The Twist knots  $\mathcal{T}_n = C(n, 2)$  are not harmonic knots  $H(4, b, c)$  for  $n > 3$ . They are not harmonic knots  $H(3, b, c)$  for  $n > 2$ .

*Proof.* The Schubert fractions of  $\mathcal{T}_n = S(n + \frac{1}{2})$  (or  $\overline{\mathcal{T}_n}$ ) with an even denominator are  $\frac{2n+1}{2}$ , and  $\frac{2n+1}{-n}$  or  $\frac{2n+1}{n+1}$  according to the parity of  $n$ . The only such fractions verifying  $\beta^2 \equiv \pm 2 \pmod{\alpha}$  are  $\frac{3}{2}, \frac{7}{4}, \frac{9}{4}$ . The first two are the Schubert fractions of the trefoil and the  $\overline{5}_2$  knot, which are harmonic for  $a = 4$ . We have only to study the case of  $6_1 = S(\frac{9}{4})$ . We have  $\frac{9}{4} = [1, 2, -1, 2, 1, -2, 1, 2]$ . Since this continued fraction expansion has two consecutive sign changes, we see that  $6_1$  is not of the form  $H(4, b, c)$ .  $\square$

But there also exist infinitely many rational knots whose Schubert fractions  $\frac{\alpha}{\beta}$  satisfy  $\beta^2 \equiv -2 \pmod{\alpha}$  that are not harmonic knots for  $a = 4$ .

**Proposition 3.11.** *The knots  $S(n + \frac{1}{2n})$  are not harmonic knots  $H(4, b, c)$  for  $n > 1$ . Their crossing number is  $3n$  and their Schubert fractions  $\frac{\alpha}{\beta} = \frac{2n^2+1}{2n}$  satisfy  $\beta^2 \equiv -2 \pmod{\alpha}$ .*

*Proof.* We shall use the Möbius transformations  $F(x) = [1, 2, x] = \frac{3x+1}{2x+1}$ ,  
 $C(x) = [1, 2, -1, 2, x] = x+2$ ,  $D(x) = [1, -2, 1, 2, x] = \frac{x}{4x+1}$ , and  $D^k(x) = \frac{x}{4kx+1}$ .  
 If  $n = 2k$ , we have  $C^k(x) = 2k+x$  and  $D^k(\infty) = \frac{1}{4k}$ , and then

$$n + \frac{1}{2n} = C^k D^k(\infty).$$

If  $n = 2k+1$ , we have  $FD^k(\infty) = F(\frac{1}{4k}) = \frac{2n+1}{2n}$ , and then

$$n + \frac{1}{2n} = n - 1 + \frac{2n+1}{2n} = C^k FD^k(\infty).$$

These continued fractions are such that  $\beta^2 \equiv -2 \pmod{\alpha}$ . Nevertheless, for  $n > 1$  these continued fractions have two consecutive sign changes, and therefore they do not correspond to harmonic knots  $H(4, b, c)$ .  $\square$

### 3.2 Proof of theorem 3.6

The crossing points of the plane projection of  $H = H(4, b, c)$  are obtained for parameter pairs  $(t, s)$  where  $t = \cos(\frac{m}{4b}\pi)$ ,  $s = \cos(\frac{m'}{4b}\pi)$ . We shall denote  $\lambda = \frac{3b-c}{4}$  (or  $c = 3b - 4\lambda$ ), and  $\theta = \frac{\lambda}{b}\pi$ . We will consider the two following cases.

**The case  $b = 4n + 1$ .**

For  $k = 0, \dots, n-1$ , let us consider the following crossing points

- $A_k$  corresponding to  $m = 4k + 1$ ,  $m' = 2b - m$ ,
- $B_k$  corresponding to  $m = 4k + 2$ ,  $m' = 4b - m$ ,
- $C_k$  corresponding to  $m = 4k + 3$ ,  $m' = 2b + m$ ,
- $D_k$  corresponding to  $m = 2b - 4(k + 1)$ ,  $m' = 4b - m$ .

Then we have

- $x(A_k) = \cos\left(\frac{4k+1}{b}\pi\right)$ ,  $y(A_k) = (-1)^k \cos \frac{\pi}{4} \neq 0$ ,
- $x(B_k) = \cos\left(\frac{4k+2}{b}\pi\right)$ ,  $y(B_k) = 0$ ,
- $x(C_k) = \cos\left(\frac{4k+3}{b}\pi\right)$ ,  $y(C_k) = (-1)^k \cos \frac{3\pi}{4} \neq 0$ ,
- $x(D_k) = \cos\left(\frac{4k+4}{b}\pi\right)$ ,  $y(D_k) = 0$ .

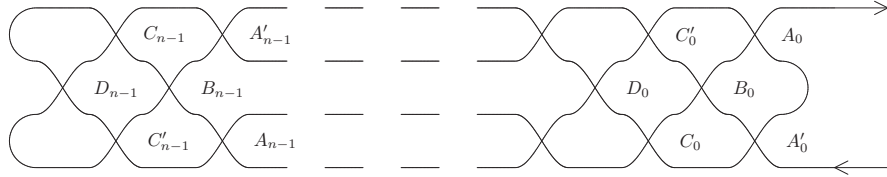


Figure 5:  $H(4, 4n + 1, c)$

Hence our  $4n$  points satisfy

$$x(A_0) > x(B_0) > x(C_0) > x(D_0) > \dots > x(A_{n-1}) > x(B_{n-1}) > x(C_{n-1}) > x(D_{n-1}).$$

Let  $A'_k$  (respectively  $C'_k$ ) be the reflection of  $A_k$  (respectively  $C_k$ ) in the  $x$ -axis. The crossings of our diagram are the points  $A_k, A'_k, B_k, C_k, C'_k$ , and  $D_k$ .

The Conway sign of a crossing point  $M$  is  $s(M) = \text{sign}(D(M))$  if  $y(M) = 0$ , and  $s(M) = -\text{sign}(D(M))$  if  $y(M) \neq 0$ .

By symmetry, we have  $s(A'_k) = s(A_k)$  and  $s(C'_k) = s(C_k)$  because symmetric points correspond to opposite parameters. The Conway form of  $H$  is then (see paragraph 2) :

$$C\left(s(D_{n-1}), 2s(C_{n-1}), s(B_{n-1}), 2s(A_{n-1}), \dots, s(B_0), 2s(A_0)\right).$$

Using the identity  $T'_a(\cos \tau) = a \frac{\sin a\tau}{\sin \tau}$ , we get  $x'(t)y'(t) \sim \sin\left(\frac{m}{b}\pi\right) \sin\left(\frac{m}{4}\pi\right)$ . Consequently,

- For  $A_k$  we have  $x'(t)y'(t) \sim \sin\left(\frac{4k+1}{b}\pi\right) \sin\left(\frac{4k+1}{4}\pi\right) \sim (-1)^k$ .

- Similarly, for  $B_k$   $C_k$  and  $D_k$  we obtain  $x'(t)y'(t) \sim (-1)^k$ .

On the other hand, at the crossing points we have

$$z(t) - z(s) = 2 \sin\left(\frac{c}{8b}(m' - m)\pi\right) \sin\left(\frac{c}{8b}(m + m')\pi\right).$$

We obtain the signs of our crossing points, with  $c = 3b - 4\lambda$ ,  $\theta = \frac{\lambda}{b}\pi$ .

- For  $A_k$  we get:  $z(t) - z(s) = 2 \sin \frac{c}{b}(n - k)\pi \sin c \frac{\pi}{4}$ .  
We have  $\sin c \frac{\pi}{4} = \sin \frac{12n + 3 - 4\lambda}{4}\pi = (-1)^{n+\lambda} \sin \frac{3\pi}{4} \sim (-1)^{n+\lambda}$   
and also  $\sin\left(\frac{c}{b}(n - k)\pi\right) = \sin\left(\left(3 - \frac{4\lambda}{b}\right)(n - k)\pi\right)$   
 $= (-1)^{n+k} \sin\left(\frac{4k - 4n}{b}\lambda\pi\right) = (-1)^{n+k+\lambda} \sin(4k + 1)\theta$

Consequently, the sign of  $A_k$  is  $s(A_k) = -\text{sign}(\sin(4k + 1)\theta)$ .

- For  $B_k$ , we have:  $z(t) - z(s) = 2 \sin\left(\frac{c}{b}(2n - k)\pi\right) \sin c \frac{\pi}{2} = -2 \sin\left(\frac{c}{b}(2n - k)\pi\right)$   
 $= 2 \sin\left(\left(3 - \frac{4\lambda}{b}\right)(k - 2n)\pi\right)$   
 $= 2(-1)^{k+1} \sin\left(\frac{\lambda}{b}(4k - 8n)\pi\right) = (-1)^{k+1} \sin(4k + 2)\theta..$

Therefore the Conway sign of  $B_k$  is  $s(B_k) = -\text{sign}(\sin(4k + 2)\theta)$ .

- For  $C_k$ :  $z(t) - z(s) = 2 \sin\left(\frac{c}{4}\pi\right) \sin\left(\frac{c}{b}(n + k + 1)\pi\right)$ .

We know that  $\sin \frac{c\pi}{4} \sim (-1)^{n+\lambda}$ . Let us compute the second factor

$$\begin{aligned} \sin\left(\left(3 - \frac{4\lambda}{b}\right)(n + k + 1)\pi\right) &= (-1)^{n+k} \sin\left(\frac{\lambda}{b}(4n + 4k + 4)\pi\right) \\ &= (-1)^{n+k} \sin\left(\frac{\lambda}{b}(b + 4k + 3)\pi\right) = (-1)^{n+k+\lambda} \sin(4k + 3)\theta. \end{aligned}$$

Hence the sign of  $C_k$  is  $s(C_k) = -\text{sign}(\sin(4k + 3)\theta)$ .

- For  $D_k$ :  $z(t) - z(s) = 2 \sin\left(\frac{c}{b}(k + 1)\pi\right) \sin\left(c \frac{\pi}{2}\right)$   
 $= -2 \sin\left(\left(3 - \frac{4\lambda}{b}\right)(k + 1)\pi\right) = (-1)^{k+1} \sin(4k + 4)\theta.$

We conclude that  $s(D_k) = -\text{sign}(\sin(4k + 4)\theta)$ .

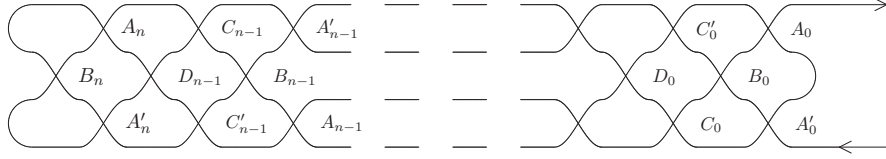
This completes the computation of our Conway normal form of  $H$  in this first case.

### The case $b = 4n + 3$ .

Here, the diagram is different. Let us consider the following  $4n + 2$  crossing points.

For  $k = 0, \dots, n$

- $A_k$  corresponding to  $m = 4k + 1$ ,  $m' = 2b + m$ ,
- $B_k$  corresponding to  $m = 4k + 2$ ,  $m' = 4b - m$ .

Figure 6:  $H(4, 4n + 3, c)$ 

For  $k = 0, \dots, n-1$

- $C_k$  corresponding to  $m = 4k + 3$ ,  $m' = 2b - m$ ,
- $D_k$  corresponding to  $m = 2b + 4(k + 1)$ ,  $m' = 4b - m$ .

These points are such that

$$x(A_0) > x(B_0) > x(C_0) > x(D_0) > \dots > x(C_{n-1}) > x(D_{n-1}) > x(A_n) > x(B_n),$$

and we have  $\text{sign}(x'(t)y'(t)) = (-1)^k$ .

- For  $A_k$  we have  $z(t) - z(s) = 2 \sin(c \frac{\pi}{4}) \sin(\frac{c}{b}(n + k + 1)\pi)$ .

We easily get  $\text{sign}(\sin c \frac{\pi}{4}) = (-1)^{n+\lambda}$ . We also obtain

$$\begin{aligned} \sin(\frac{c}{b}(n + k + 1)\pi) &= \sin\left(\left(3 - \frac{4\lambda}{b}\right)(n + k + 1)\pi\right) \\ &= (-1)^{n+k} \sin\left(\frac{\lambda}{b}(b + 4k + 1)\pi\right) = (-1)^{n+k+\lambda} \sin(4k + 1)\theta. \end{aligned}$$

Hence the sign of  $A_k$  is  $s(A_k) = -\text{sign}(\sin(4k + 1)\theta)$ .

- For  $B_k$  we have  $z(t) - z(s) = 2 \sin(\frac{c}{b}(2n + 1 - k)\pi) \sin c \frac{\pi}{2}$ .

We have  $\sin(c \frac{\pi}{2}) = 1 > 0$ , and

$$\begin{aligned} \sin(\frac{c}{b}(2n + 1 - k)\pi) &= \sin\left(\left(3 - \frac{4\lambda}{b}\right)(2n + 1 - k)\pi\right) \\ &= (-1)^{k+1} \sin\left(\frac{\lambda}{b}(4k - 8n - 4)\pi\right) = (-1)^{k+1} \sin(4k + 2)\theta. \end{aligned}$$

Then, the sign of  $B_k$  is  $s(B_k) = -\text{sign}(\sin(4k + 2)\theta)$ .

- For  $C_k$  we have  $z(t) - z(s) = 2 \sin(\frac{c}{b}(n - k)\pi) \sin c \frac{\pi}{4}$ .

We obtain

$$\begin{aligned} \sin(\frac{c}{b}(n - k)\pi) &= \sin\left(\left(3 - \frac{4\lambda}{b}\right)(n - k)\pi\right) \\ &= (-1)^{n+k} \sin\left(\frac{4k - 4n}{b} \lambda \pi\right) = (-1)^{n+k+\lambda} \sin(4k + 3)\theta. \end{aligned}$$

The sign of  $C_k$  is then  $s(C_k) = -\text{sign}(\sin(4k + 3)\theta)$ .

- For  $D_k$  we have  $z(t) - z(s) = 2 \sin(-\frac{c}{b}(k+1)\pi) \sin c \frac{\pi}{2}$ . We have  $\sin c \frac{\pi}{2} > 0$ . We also have

$$\sin(-\frac{c}{b}(k+1)\pi) = \sin\left(\left(\frac{4\lambda}{b} - 3\right)(k+1)\pi\right) (-1)^{k+1} \sin(4k+4)\theta.$$

Consequently, the sign of  $D_k$  is  $s(D_k) = -\text{sign}(\sin(4k+4)\theta)$ .

This concludes the computation of the Conway normal form of  $H(4, b, c)$ . In both cases it is  $C(e_1, 2e_2, \dots, e_{b-2}, 2e_{b-1})$  where  $e_j = \text{sign}(\sin(j-b)\theta)$ .

If  $b < c < 3b$ , then we get  $\lambda < \frac{b}{2}$ , and  $\theta < \frac{\pi}{2}$ . Consequently, there is no two consecutive sign changes in our sequence. Furthermore, the total number of sign changes is  $\lambda - 1$ . We conclude by Proposition 2.2 that the crossing number is  $N = \frac{3(b-1)}{2} - (\lambda - 1) = \frac{3b + c - 2}{4}$ . The fact that  $\beta^2 \equiv \pm 2 \pmod{\alpha}$  is a consequence of Proposition 2.6.  $\square$

## 4 Some families with $a \geq 5$

We will consider Chebyshev curves as trajectories in a rectangular billiard (see [KP3]).

**Lemma 4.1.** *Let  $C(t)$  be the plane curve parametrized by  $x(t) = T_a(t)$ ,  $y(t) = T_b(t)$ , and let  $F$  be the function defined by  $F(x) = \frac{2}{\pi}(\arccos(x) - 1)$ . The mapping  $(x, y) \mapsto (bF(x), aF(y))$  is an homeomorphism from the square  $I = (-1, 1)^2$  onto the rectangle  $(-b, b) \times (-a, a)$ . The curve  $B(t)$  obtained from  $C(t)$  is a “billiard trajectory”. The slopes of its segments are  $\pm 1$ .*

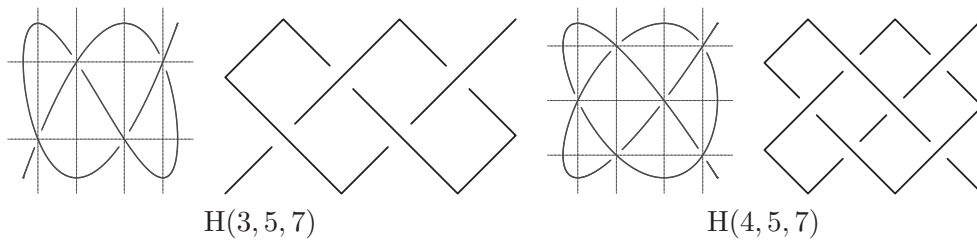
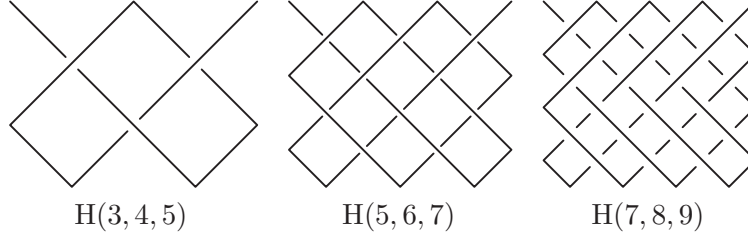


Figure 7: Billiard representations of  $4_1$  and  $5_2$

### 4.1 The harmonic knots $H(2n-1, 2n, 2n+1)$

Let us begin with some simple observations on the diagram of  $K_n = H(2n-1, 2n, 2n+1)$ .

We have  $z(t) = 2t \cdot y(t) - x(t)$ . Consequently, if  $(t, s)$  is a parameter pair corresponding to a crossing, we have:  $z(t) - z(s) = 2(t-s)y(t)$ . This simple rule allows us to draw by hand the billiard picture of the knot  $K_n$  (see Figure 8):

Figure 8: The knots  $K_n$  for  $n = 2, 3, 4$ 

We can even deduce a simpler rule as follows.

**Lemma 4.2.** *Let  $K = H(a, b, c)$  with  $b = a + 1$ . Then the sign of a crossing of parameters  $(s, t)$  is  $\text{sign}(D) = \text{sign}((z(t) - z(s))(t - s))$ .*

*Proof.* Let  $(s, t)$  be the parameter pair of a crossing. We have

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi, \quad s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi, \quad 0 < \frac{k}{a} + \frac{h}{b} < 1.$$

An easy calculation shows that, when  $b = a + 1$  then

$$x'(t)y'(t) \sim -\sin\left(\frac{k}{a}\pi\right) \sin\left(\frac{h}{b}\pi\right) \sim t - s,$$

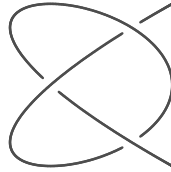
which concludes the proof, using Lemma 3.2.  $\square$

**Corollary 4.3.** *The sign of a crossing of  $H(2n - 1, 2n, 2n + 1)$  is  $\text{sign}(D) = \text{sign}(y)$ .*

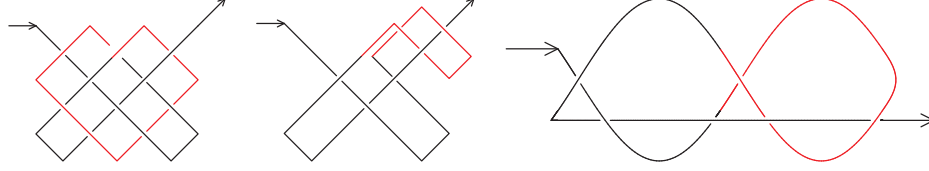
**Remark 4.4.** This result is still true when  $(a, b, c)$  are in a more general arithmetic progression (see Figures 25 or 24).

**Theorem 4.5.** *The knot  $H(2n - 1, 2n, 2n + 1)$  is isotopic to  $H(4, 2n - 1, 2n + 1)$  if  $n$  is odd, and to  $H(4, 2n + 1, 2n - 1)$  if  $n$  is even. Its crossing number is  $2n - 1$ .*

*Proof.* Our proof is by induction on  $n$ . We shall prove that  $K_n$  is isotopic to the two-bridge knot of Conway form  $C(1, 2, (-1)^1, 2(-1)^1, \dots, (-1)^{n-2}, 2(-1)^{n-2})$ .

Figure 9: The knot  $K_2$  is a trefoil

For  $n = 2$ , the knot  $H(3, 4, 5)$  is the trefoil  $K_2 = C(1, 2) = \overline{H}(4, 3, 5)$ .

Figure 10: An isotopy of  $K_3$ 

For  $n = 3$ , Figure 10 shows that  $K_3 = C(1, 2, -1, -2)$ . It also gives an idea of our proof.

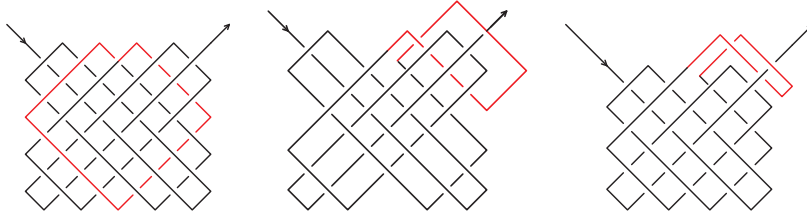
By induction, let us suppose that  $K_{n-1} = C(1, 2, (-1)^1, (-1)^1 2, \dots, (-1)^{n-3}, (-1)^{n-3} 2)$ . We shall consider  $K_n$  to be composed of two parts.

The first part  $L$  is a loop (the red loop of Figure 11) which is symmetrical about the  $y$ -axis, and consists of the points of parameters  $t \in I = (\pi(\frac{1}{2} - \frac{1}{2n-1}), \pi(\frac{1}{2} + \frac{1}{2n-1}))$ . It contains exactly  $2(2n - 3)$  crossing points, which are the points of parameters

$$t = \cos \tau, \quad \tau = \frac{\pi}{2} + \frac{k\pi}{2n(2n-1)}, \quad |k| \leq 2n-2, \quad k \neq 0, \pm n.$$

The other part  $K'$  consists of the points of parameters  $t \in \mathbf{R} - I$ .

When  $n$  is odd, the part of the loop  $L$  where  $t < \frac{\pi}{2}$  is over the bounded part of  $K'$ , and the other part of  $L$  is under the bounded part of  $K'$ . When  $n$  is even, the first part of  $L$  is under and the second part of  $L$  is over the bounded part of  $K'$ . Consequently, it is possible

Figure 11: Pulling the loop  $L$  away from  $K_n$ , we obtain  $K_{n-1}$ .

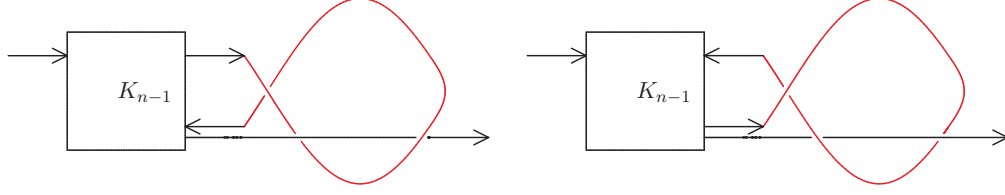
to move the loop  $L$  away from the bounded part of  $K'$ , and we see that  $K_n$  is obtained from  $K'$  by a weaving process (see [Ka, p. 50]).

Now let us look at the billiard drawing of  $K'$ . It is clear (see Figure 11) that, inside the rectangle  $|X| \leq 2n-1, |Y| \leq 2n-2$ , the diagram of  $K'$  coincides with the billiard diagram of  $K_{n-1}$ .

Consequently, our weaving are illustrated in figure 12.

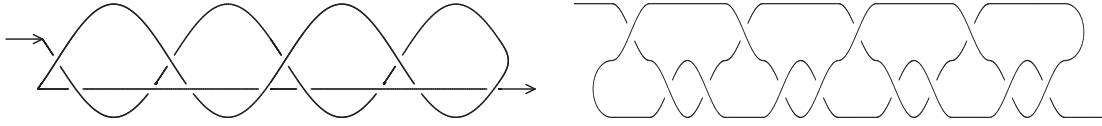
If  $n$  is even, then using the induction hypothesis, we obtain the Conway form  $K_n = C(1, 2, -1, -2, \dots, 1, 2)$  of length  $2n-2$ . If  $n$  is odd, then we obtain the Conway form  $K_n = C(1, 2, -1, -2, \dots, -1, -2)$  of length  $2n-2$ . This completes our induction proof.



Figure 12: The weaving process:  $n$  odd (left),  $n$  even (right)

By the proof of corollary 3.9, we deduce that  $K_n$  is isotopic to  $H(4, 2n - 1, 2n + 1)$  if  $n$  is odd, and to  $H(4, 2n + 1, 2n - 1)$  if  $n$  is even.  $\square$

The result of this inductive weaving process is illustrated in Figure 13 for the knot  $K_5$ .

Figure 13: The knot  $K_5$  is a two-bridge knot

#### 4.2 The harmonic knots $H(5, k, k + 1)$ .

The bridge number of such a knot is at most three, and there is no obvious reason for it to be smaller. This is why the following result surprised us.

**Theorem 4.6.**

*The knot  $H(5, 5n + 1, 5n + 2)$  is the two-bridge knot of Conway form  $C(2n + 1, 2n)$ .*

*The knot  $H(5, 5n + 3, 5n + 4)$  is the two-bridge knot of Conway form  $C(2n + 1, 2n + 2)$ .*

*Besides  $H(5, 6, 7) = H(4, 5, 7)$  and  $H(5, 3, 4)$ , these knots are not of the form  $H(a, b, c)$  with  $a \leq 4$ .*

*Proof.* We shall often need the toric move shown in Figure 14. The equivalence of the two

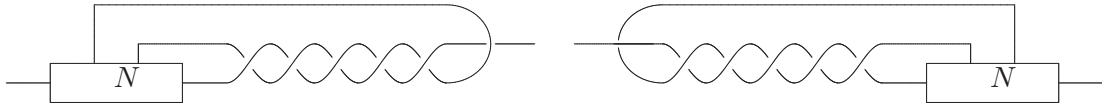
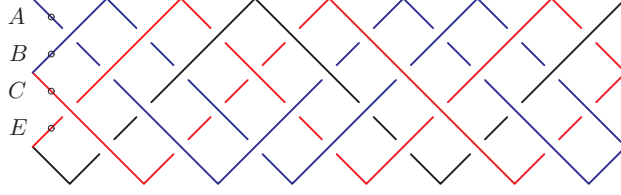


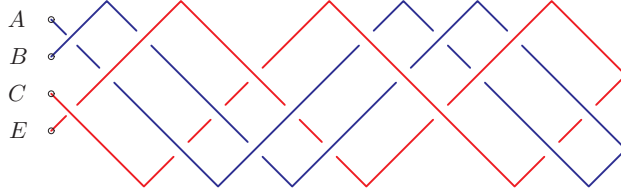
Figure 14: The toric move

diagrams is clear if we consider the compactification in  $S^3$  of this knot.

First, we shall study the knots  $H_n = H(5, 5n + 1, 5n + 2)$ . We shall provide drawings of an isotopy showing that  $H_3 = C(7, 6)$ , and show how the different steps generalize when  $n \neq 3$ .

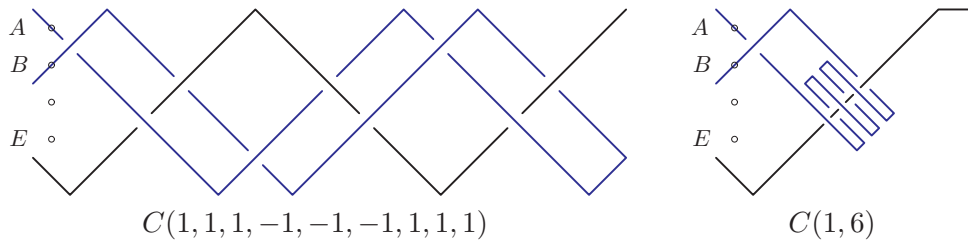
Figure 15: The knot  $H_3 = H(5, 16, 17)$ 

Let us introduce the points  $A, B, C, E$  as shown in Figure 15. We shall consider the knot to be divided in three parts: the loop  $\alpha$  from  $A$  to  $B$ , the loop  $\beta$  from  $C$  to  $E$ , the "stick"  $\gamma$  is the rest of the knot.

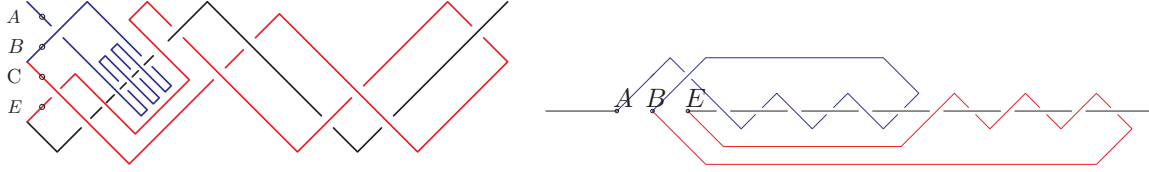
Figure 16: The loops  $\tilde{\alpha}$  and  $\tilde{\beta}$  are unlinked

First, we notice that the loops  $\tilde{\alpha} = \alpha \cup [A, B]$  and  $\tilde{\beta} = \beta \cup [C, E]$  are unlinked.

This is clear when  $n = 3$  (see Figure 16), and can be proved by induction for the general case. It is also possible to deduce this from the fact that  $\tilde{\alpha} \cup \tilde{\beta}$  is a 2-bridge link of Schubert fraction 0, because its Conway normal form is  $C(0, 1, 0, -1, \dots, 0, -1)$ . Using this fact, we can shrink  $\alpha$  towards the left as shown in Figure 17.

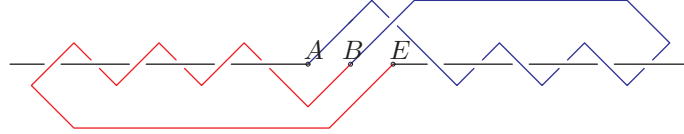
Figure 17:  $\alpha$  is the rational tangle of Conway form  $C(1, 1, 1, -1, -1, -1, 1, 1, 1)$ 

Then we simplify it using the fact that the tangle  $\alpha$  is the rational tangle of Conway form  $C(1, 1, 1, -1, -1, -1, 1, 1, 1)$ , and consequently is isotopic to the tangle  $C(1, 6)$ . In the general case, this tangle is isotopic to  $C(1, 2n)$ . Now, our knot resembles Figure 18 (left). The right part of the knot is a tangle of Conway form  $C(2, 1, -1, -1, -1, 1, 1)$  and then it is isotopic to the tangle  $C(6)$ . Using this isotopy, we obtain the diagram of figure 18 (right). In the general case this right part is of Conway form  $C(1, 2, x)$  where

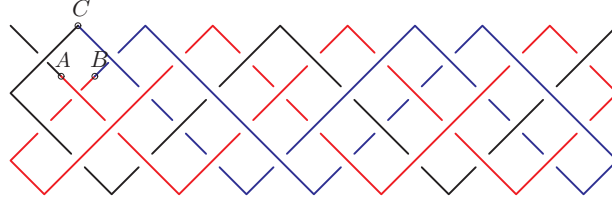
Figure 18: A simplified version of the knot  $H_3$ 

$x = (-1, -1, -1, 1, 1, 1, \dots, (-1)^{n-2}, (-1)^{n-2}, (-1)^{n-2}, (-1)^{n-1}, (-1)^{n-1})$  and is isotopic to  $C(2n)$ .

Finally we slide the loop  $\beta$  from right to left by a toric move. The resulting diagram is shown in Figure 19; it is of Conway form  $C(7, 6)$  ( $C(2n + 1, 2n)$  in the general case).

Figure 19: The knot  $H_3 = H(5, 16, 17)$  is isotopic to the knot of Conway form  $C(7, 6)$ 

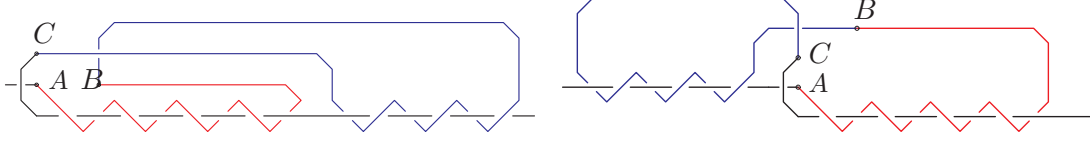
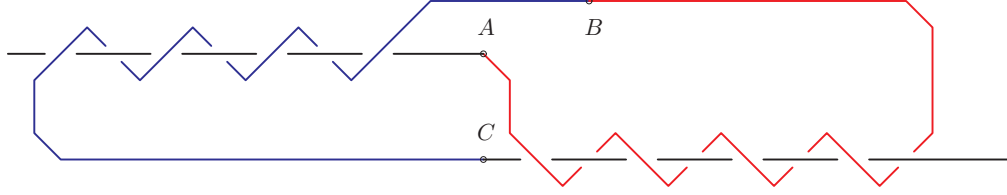
The study of the knots  $H(5, 5n + 3, 5n + 4)$  is similar; we only give a few figures showing the isotopy  $H(5, 18, 19) = C(7, 8)$ . First, we consider the points  $A, B, C$  as shown in Figure

Figure 20: The knot  $H(5, 18, 19)$ 

20. The knot is composed of three parts: the loop  $\alpha$  from  $A$  to  $B$ , the loop  $\beta$  from  $B$  to  $C$ , and the rest of the knot.

As before, the loops  $\alpha$  (from  $A$  to  $B$ ) and  $\beta$  (from  $B$  to  $C$ ) are unlinked, and therefore  $\alpha$  can be shrunk towards the left and simplified. The loop  $\beta$  is simplified too, and we obtain the diagram shown in Figure 21 (left). Applying a toric move to the loop  $\beta$ , we obtain the diagram of Figure 21 (right). The last move is easy to see, we simply pull the half-loop containing  $C$  downwards. The resulting diagram shown in Figure 22 is of Conway form  $C(7, 8)$ .

Except for  $H(5, 6, 7)$  and  $H(5, 3, 4)$ , these knots do not have any Schubert fraction  $\frac{\alpha}{\beta}$  such that  $\beta^2 \equiv \pm 1$  or  $\beta^2 \equiv \pm 2 \pmod{\alpha}$ ; therefore they are not harmonic for  $a \leq 4$ .  $\square$

Figure 21: Applying a toric move to the loop  $\beta$ Figure 22: The knot  $H(5, 18, 19)$  is isotopic to the knot of Conway form  $C(7, 8)$ 

### 4.3 Some new findings on harmonic knots

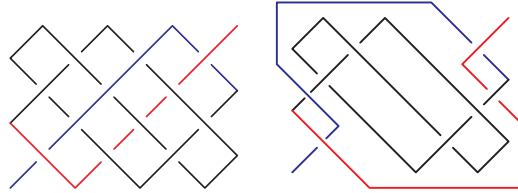
Thanks to the simplicity of our billiard diagrams, we can easily compute the Alexander polynomials of our knots (see [Li]). On the other hand, there is a list of the Alexander polynomials of the first prime knots with 15 or fewer crossings in [KA].

Using some evident simplifications, we can estimate the crossing number  $N$  and identify the knot.

We first give some specific examples, then an exhaustive list of harmonic knots  $H(a, b, c)$  with  $(a - 1)(b - 1) \leq 30$ . Their diagrams have 15 or fewer crossing points.

#### Harmonic knots are not necessarily prime.

G. and J. Freudentburg conjectured that every harmonic knot is prime. This conjecture is not true. The knot  $H(5, 7, 11)$  is not prime; it is the connected sum of two figure-eight knots.

Figure 23: The knot  $H(5, 7, 11)$  is composite

#### Harmonic knots may be nonreversible.

We have identified the following knots of form  $H(2n - 1, 2n + 1, 2n + 3)$ ,  $n \geq 4$ , by computing their Alexander polynomials and estimating their crossing numbers. We obtain

$$\begin{aligned} H(3, 5, 7) &= 4_1, & H(5, 7, 9) &= H(3, 7, 11) = 6_3, & H(7, 9, 11) &= H(5, 9, 13) = 8_{17}, \\ H(9, 11, 13) &= H(7, 11, 15) = 10_{115}, & H(11, 13, 15) &= H(9, 13, 17) = 12a_{1167}. \end{aligned}$$

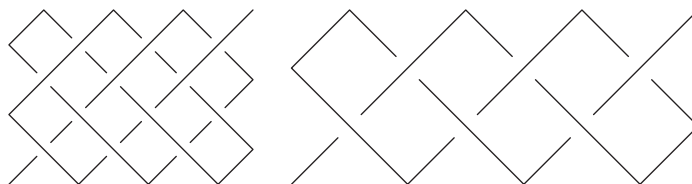
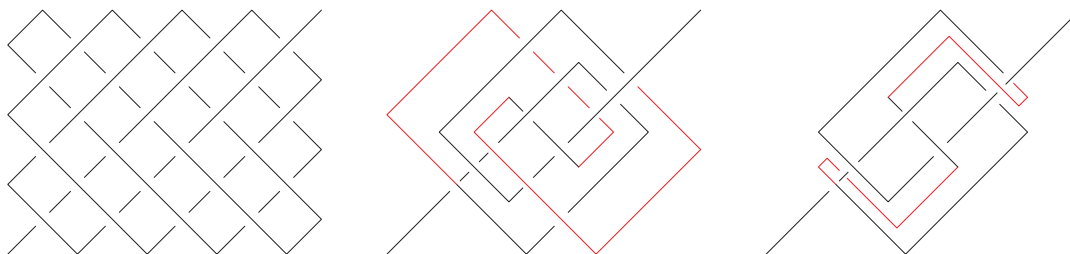
Figure 24: The knots  $H(5, 7, 9)$  and  $H(3, 7, 11)$  are isotopic to  $6_3$ 

Figure 25 shows that  $H(7, 9, 11) = 8_{17}$  is symmetrical about the origin and therefore is strongly  $(-)$ amphicheiral. It is also the first nonreversible knot (see also [Cr, p. 30]).

Figure 25: The knot  $H(7, 9, 11)$ , an unusual model of  $8_{17}$ 

### A table of harmonic knots with $(a - 1)(b - 1) \leq 30$ .

Here, we provide a table giving the Alexander polynomial of the harmonic knots with diagrams having 15 or fewer crossings, that is  $H(a, b, c)$  with  $3 \leq a < b < c$ ,  $(a-1)(b-1) \leq 30$  and  $(c, ab) = 1$ . Using Lemma 3.5, we choose  $c$  such that  $c \neq \lambda a + \mu b$ ,  $\lambda, \mu > 0$  (see also [FF]). We have to consider 51 different harmonic knots.

In cases where  $a = 3$  or  $a = 4$ , we know that  $H(a, b, c)$  is a two-bridge knot. The crossing number of such a knot is  $\frac{1}{3}(b + c)$ , when  $a = 3$  and  $\frac{1}{4}(3b + c - 2)$  when  $a = 4$ . Furthermore, its Schubert fraction is computed using Theorem 3.7 or [KP4, Th. 6.5].

When  $a \geq 5$ , we compute the Alexander polynomial of the knot and compare it with the tables. Sometimes (when starred) it is also necessary to use Knotscape ([KS]) to determine the name of the knot.

| Table of the the first harmonic knots |          |          |                |          |           |
|---------------------------------------|----------|----------|----------------|----------|-----------|
|                                       | Fraction | Name     |                | Fraction | Name      |
| $H(3, 4, 5)$                          | 3        | $3_1$    | $H(3, 5, 7)$   | $5/2$    | $4_1$     |
| $H(3, 7, 8)$                          | 5        | $5_1$    | $H(3, 7, 11)$  | $13/5$   | $6_3$     |
| $H(3, 8, 13)$                         | $21/8$   | $7_7$    | $H(3, 10, 11)$ | 7        | $7_1$     |
| $H(3, 10, 17)$                        | $55/21$  | $9_{31}$ | $H(3, 11, 13)$ | $17/4$   | $8_3$     |
| $H(3, 11, 16)$                        | $39/14$  | $9_{17}$ | $H(3, 11, 19)$ | $89/34$  | $10_{45}$ |

|            |         |                   |            |         |                                |
|------------|---------|-------------------|------------|---------|--------------------------------|
| H(3,13,14) | 9       | 9 <sub>1</sub>    | H(3,13,17) | 53/23   | 10 <sub>28</sub>               |
| H(3,13,20) | 105/41  | 11a175            | H(3,13,23) | 233/89  | 12a499                         |
| H(3,14,19) | 77/34   | 11a119*           | H(3,14,25) | 377/144 | 13a1739                        |
| H(3,16,17) | 11      | 11a367            | H(3,16,23) | 187/67  | 13a2124*                       |
| H(3,16,29) | 987/377 | 15a39533*         | H(4,5,7)   | 7/2     | 5 <sub>2</sub>                 |
| H(4,5,11)  | 11/3    | 6 <sub>2</sub>    | H(4,7,9)   | 17/5    | 7 <sub>5</sub>                 |
| H(4,7,13)  | 23/5    | 8 <sub>7</sub>    | H(4,7,17)  | 41/11   | 9 <sub>20</sub>                |
| H(4,9,11)  | 41/12   | 9 <sub>18</sub>   | H(4,9,19)  | 89/25   | 11a180                         |
| H(4,9,23)  | 153/41  | 12a541            | H(4,11,13) | 99/29   | 11a236                         |
| H(4,11,17) | 113/31  | 12a758            | H(4,11,21) | 187/41  | 13a2679*                       |
| H(4,11,25) | 329/87  | 14a7552*          | H(4,11,29) | 571/153 | 15a42637*                      |
| H(5,6,7)   | 7/4     | 5 <sub>2</sub>    | H(5,6,13)  |         | 10 <sub>159</sub>              |
| H(5,6,19)  |         | 10 <sub>116</sub> | H(5,7,8)   | 5/2     | 4 <sub>1</sub>                 |
| H(5,7,9)   | 13/8    | 6 <sub>3</sub>    | H(5,7,11)  |         | 4 <sub>1</sub> #4 <sub>1</sub> |
| H(5,7,13)  |         | 12n356            | H(5,7,16)  |         | 12n798                         |
| H(5,7,18)  |         | 12n321            | H(5,7,23)  |         | 12a960                         |
| H(5,8,9)   | 13/4    | 7 <sub>3</sub>    | H(5,8,11)  | 21/13   | 7 <sub>7</sub>                 |
| H(5,8,17)  |         | 14n22712*         | H(5,8,19)  |         | 14n26120*                      |
| H(5,8,27)  |         | 14a19221*         | H(6,7,11)  |         | 10 <sub>134</sub>              |
| H(6,7,17)  |         | 15n42918*         | H(6,7,23)  |         | 15n165258*                     |
| H(6,7,29)  |         | 15a81117          |            |         |                                |

### Some isotopic harmonic knots

Theorem 4.5 asserts that the knots  $H(2n-1, 2n, 2n+1)$  and  $H(4, 2n-1, 2n+1)$  are isotopic and therefore can be identified.

Theorem 4.6 asserts that the knots  $H(5, 5k+1, 5k+2)$  and  $H(5, 5k+3, 5k+4)$  are also two-bridge knots. The knots  $H(5, 5k+2, 5k+3)$ ,  $2 \leq k \leq 8$  are not two-bridge knots because their modulo 2 Conway polynomials are not Fibonacci polynomials (see [KP5]). Note that  $H(5, 7, 8)$  is the figure-eight knot  $4_1$ .

We have observed that for some values  $(a, b, c)$  in arithmetic progression,  $H(b-k, b, b+k) = H(b-\lambda k, b, b+\lambda k)$ , for some  $\lambda > 1$ . It happens for example with  $H(5, 11, 17) = H(9, 11, 13)$ ,  $H(3, 11, 19) = H(7, 11, 15)$ ,  $H(5, 9, 13) = H(7, 9, 11)$ , and many others. It would be interesting to explain this phenomenon.

### References

- [KA] D. Bar Nathan, S. Morrison, *Knot Atlas*, Oct. 2010,  
[http://katlas.org/wiki/The\\_Take\\_Home\\_Database](http://katlas.org/wiki/The_Take_Home_Database).

- [BDHZ] A. Boocher, J. Daigle, J. Hoste, W. Zheng, *Sampling Lissajous and Fourier knots*, Exp. Math. **18**(4), 481-497 (2009).
- [BHJS] M. G. V. Bogle, J. E. Hearst, V. F. R. Jones, L. Stoilov, *Lissajous knots*, Journal of Knot Theory and its Ramifications, 3(2): 121-140, (1994).
- [Com] E. H. Comstock, *The Real Singularities of Harmonic Curves of three Frequencies*, Trans. of the Wisconsin Academy of Sciences, Vol XI : 452-464, (1897).
- [Con] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), 329-358 Pergamon, Oxford (1970)
- [Cr] P. R. Cromwell, *Knots and links*, Cambridge University Press, Cambridge, 2004. xviii+328 pp.
- [Fi] G. Fischer, *Plane Algebraic Curves*, A.M.S. Student Mathematical Library Vol 15, 2001.
- [FF] G. Freudenburg, J. Freudenburg, *Curves defined by Chebyshev polynomials*, 19 p., (2009), [arXiv:0902.3440](#)
- [HZ] J. Hoste, L. Zirbel, *Lissajous knots and knots with Lissajous projections*, Kobe Journal of mathematics, vol 24, n°2, 2007.
- [JP] V. F. R. Jones, J. Przytycki, *Lissajous knots and billiard knots*, Banach Center Publications, 42:145-163, (1998).
- [Ka] L. Kauffman, *On Knots* (Princeton University Press, 1987).
- [KP1] P. -V. Koseleff, D. Pecker, *On polynomial Torus Knots*, Journal of Knot Theory and its Ramifications, Vol. **17** (**12**) (2008), 1525-1537.
- [KP3] P. -V. Koseleff, D. Pecker, *Chebyshev knots*, Journal of Knot Theory and its Ramifications, Vol. **20** (4) (2011), 575-593
- [KP4] P. -V. Koseleff, D. Pecker, *Chebyshev diagrams for two-bridge knots*, Geom. Dedicata **150** (1) , (2011), 405-425.
- [KPR] P. -V. Koseleff, D. Pecker, F. Rouillier, *The first rational Chebyshev knots*, J. Symb. Comput. **45**(12), (2010), 1341-1358.
- [KP5] P. -V. Koseleff, D. Pecker, *Conway polynomial of two-bridge links*, [arXiv:1011.5992](#)
- [KS] J. Hoste, M. Thistlethwaite, *Knotscape*, <http://www.math.utk.edu/~morwen/knotscape.html>
- [La1] C. Lamm, *There are infinitely many Lissajous knots*, Manuscripta Math., 93: 29-37, (1997).
- [La2] C. Lamm *Zylinder-Knoten und symmetrische Vereinigungen*, Dissertation, Universität Bonn, Mathematisches Institut, Bonn, 1999.
- [Li] Livingston, C., *Knot Theory*, Washington, DC: Math. Assoc. Amer., 1993.

- [Mu] K. Murasugi, *Knot Theory and its Applications*, Boston, Birkhäuser, 341p., 1996.
- [P1] D. Pecker, *Simple constructions of algebraic curves with nodes*, Compositio Math. 87 (1993), no. 1, 1–4.